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Deterministic and stochastic optimization of composite cylindrical laminates

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Abstract

The paper is focused on optimization of prestress and placement of fibers in laminated cylindrical composites. It also involves a stochastic study of prestress deviation in particular layers. Optimization (design) parameters considered in control of internal stresses are the eigenstrains. The behavior of a certain functional serving for optimization of the eigenstrains with stochastically perturbed and correlated values in a laminated cylindrical structure is examined. In the first part, a deterministic optimization of composite laminated cylinders is performed by means of the eigenstrains produced in the layers during the fabrication process. Because fabrication of laminates is sensitive to deviation of eigenstrain magnitudes, as shown from stochastic study, an additional minimization of the eigenstrains is introduced.

Keywords: Optimization; TFA; Deterministic solution; Stochastic solution; Laminated cylinders; Composites

1. Introduction

One of the applications of composite materials is in structures subject primarily to compressive loads, such as submersibles (Dvorak and Procházka, 1996). The incentive is the relatively high compressive strength that has been found in thick samples, e.g., 1440 MPa (209 ksi) in AS4/3501-6 carbon–epoxy system (Daniel and Isahi, 1994). More frequently reported magnitude of about 700 MPa for the system is attractive.

Both strength and stiffness of composite structures depend on the phase volume fractions, orientation of reinforcing fibers, and stacking sequences. Moreover, these quantities can be influenced by high residual stresses due to the fabrication process during construction of composite structure. The fiber prestress during fabrication may be an important source, which makes it possible to control internal stresses by a properly selected prestress distribution.

This fact is introduced into the selected computational model by virtue of eigenstrains (or internal strains), or eigenstresses (or internal stresses), see Dvorak and Procházka (1996). The idea stems from an

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original paper by Dvorak (1992), where a theory called transformation field analysis (TFA) was established. It starts with a priori stress–eigenstress or strain-eigenstrain relation for determining influence tensors *used* in the homogenization of material properties of composites. Similarly, in our study, the overall stress or overall strain field is expressed as a linear hull of the appropriate quantities (say stresses) due to external loading and the same quantities due to the eigenparameters. The paper (Dvorak, 1992) will be referred not only in optimal prestressing of laminated structures, but also in many other research fields and applications, as is pointed out in the sequel.

This paper outlines the deterministic solution of the problem of optimal prestress in laminas for stress-free states in individual laminas (and consequently strain-free states) or an optimal distribution of stresses (removal of unpleasant peaks of stresses) (for details see the works by Dvorak and Prochazka). The main goal is to introduce the stochastic solution of the problem. Two main reasons of this computational concept are:

- Undesirable errors of prestressing may be caused by the woven roving, which creates the ply directions and prestressing.
- The fibers are shifted to the inner side of the lamina, which is being prestressed, although we suppose a uniform distribution of prestressing throughout the thickness of this lamina.

There are other additional errors of less significance (change of temperature, human factor, etc.). The procedure described herein should suggest how to check whether the bearing capacity of the laminate cylinder does not increase after prestressing in comparison to non-prestressed state.

In papers (Dvorak and Procházka, 1996; Dvorak et al., 1999; Srinivas et al., 1999) the optimal prestress analysis is principally focused on submersibles; cylindrical laminates are considered. The generalized plane strain is used. The paper by Carvelli et al. (2001) generalized the shape of such structures. It is concerned with the structural response of a composite shell structure intended as a model of an underwater vehicle for service in sea environment. The main objective of the research is the prediction of the collapse pressure using both analytical expressions and linear or non-linear numerical analysis and the following comparison with the experimental pressure obtained in off-shore tests. The structure is composed of three basic parts with regular geometry: a cylindrical part and two conical and spherical end-closures with the same thickness. The cylindrical shell was made up of 7 plies of E-glass woven roving with polyester resin. Various structural analyses were conducted before performing the experiment in the sea to verify the reliability of the analytical and numerical tools.

In (Suvorov and Dvorak, 2001), an analytical procedure is described for evaluation of the effect of release of fiber prestress, applied prior to matrix consolidation, on stress distribution in individual plies and at free edges of laminated composite plates. Both thermal changes, piecewise uniform transformation strains in the plies and overall mechanical loads can be considered in the analysis. The thermal and transformation load contributions are decomposed into superpositions of certain uniform fields with mechanical loads. Release of fiber prestress is regarded as an equivalent uniaxial compression applied at the edges of each prestressed ply. Optimized distributions of fiber prestress are found in individual plies such that stresses in both laminate interior and at the free edges remain within allowable limits, while the applied mechanical load may change from zero to a certain maximum value. Specific results are found for cross-ply and quasi-isotropic symmetric S-glass/epoxy laminates under tension. They clearly demonstrate the substantial potential of fiber prestress in damage control and prevention in laminated composite structures.

An extension of the original paper by Dvorak (1992) can be found in Dvorak and Srinivas (1999). A complete formal similarity of the standard techniques for estimating overall elastic properties and phase averages of local mechanical and transformation stress and strain fields in heterogeneous materials is established for the self-consistent and Mori–Tanaka methods, and for Walpole's formulation of the Hashin–Shtrikman variational bounds. Regardless of the dissimilar and often heuristic assumptions that had

motivated the original formulations, the only essential difference between these techniques had been the choice of the comparison medium where each phase was assumed to be embedded in the solution of the inclusion problem. Therefore, any number of related averaging methods can be developed from other choices made such that the resulting moduli predictions do not violate the bounds. Several new types of admissible comparison media are proposed and evaluated for composites reinforced by spherical particles or aligned fibers and also for porous media with spherical cavities.

The choice of the appropriate comparison media seems to be of a great interest to researchers, which are concerned with numerical problems from the field of composite material structures. In Procházka and Šejnoha (1996), extended Hashin–Strikman variational principles involving the eigenparameters are derived and formulated. Moreover, the boundary integral formulation is presented to compute the influence tensors either for estimates of bounds of material properties, or for the use in optimization processes.

Paper by Chaboche et al. (2001) deals with formulation of variational principles based on the TFA. They involve couple of possibilities to formulate physically non-linear problems. The properties of eigenparameters are used: the eigenstrain can stand for plastic strain and the eigenstress can be substituted by relaxation stress.

In this paper we work with eigenparameters, which are linked with prestress forces, their directions and magnitudes. The procedure is fully described in Dvorak and Procházka (1996), Dvorak et al. (1999) and Srinivas et al. (1999). The relation between the overall stresses in the composite structure and the eigenstresses/eigenstrains provide a tool for optimization of bearing capacity of the laminated cylinders, their creep, time delay in construction of large structures, annihilation of stresses in particular layers, etc. In Appendix A, basic properties of the functionals in optimization problems discussed in this paper are shown on a simple structure.

The previous studies have also been carried out to obtain either the optimal fiber angles for the maximum stiffness or the optimal eigenstrains for the minimum internal stresses. Those studies assume deterministic conditions where both the material parameters and the applied loading have no random variations. However, the optimal eigenstrains as well as the optimal angles of fibers in the lay-up obtained under stochastic conditions may differ markedly from the deterministic solutions.

Recently, in Procházka and Náprstek (1995), optimal eigenstrains or optimal eigenstresses were solved under assumption that these quantities had stochastically perturbed and *non-correlated* values. In this paper the optimal eigenstrains are sought. The cost functional similar to that in Procházka and Náprstek (1995), should attain its minimum under assumption that the eigenstrains have stochastically perturbed and *correlated values*.

The cost functional input parameters may describe geometry and physical properties of the structure and also some additional properties. Non-unique solution of eigenstrains is expected in both deterministic and stochastic formulation. In stochastic formulation, moreover, the inputs are given with some deviations from the ideal situation required in the design stage, and the optimum of the cost functional is attained only with certain probability. Special forms of the cost functionals are studied. The correlation is expressed by a kind of distance function (as recommended by Cramér (1945) and is mostly used in applications till today) and alternatively by an exponential function. The distance function expresses in a relatively simple way correlation between all prestressed fibers (wires). It is very appropriate for this type of physical problems.

The numerical approach to both deterministic and stochastic solutions of the problem of optimal distribution of eigenstrains in laminated cylindrical composite structures is described. It deals with a minimization of hoop and axial stresses through the structure. A discussion on examples of deterministic and stochastic approaches with correlated eigenstrains is presented.

The relations between overall stresses and eigenstrains/eigenstresses can generally be derived by FEM or BEM. In order to explain the behavior of the system we use a simplified axially-symmetric layered hollow cylinder, depicted in Fig. 1(a), and the generalized plane strain, i.e., the axial deformation is equal to a certain constant, but generally does not vanish.

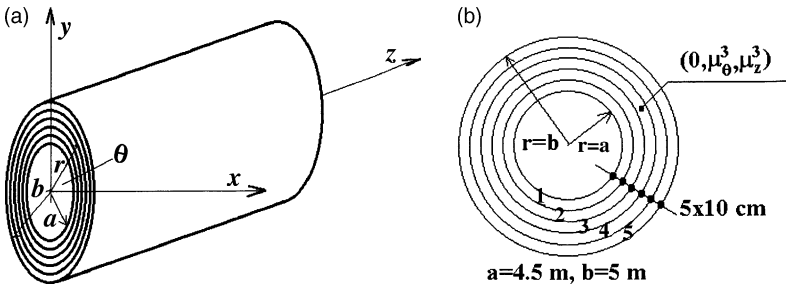

Fig. 1. (a) Geometry of the body under consideration. (b) Cross-section the structure.

As shown in Fig. 1(a), substituting the coordinate system $0x_1x_2x_3 \equiv 0xyz$ by the cylindrical system $0r\theta z$ yields simplification of definitions of our forthcoming problems. There is one-to-one mapping between these systems defined by well known formulas. In Fig. 1(a) $r = a$ is the radius of the internal surface of the hollow cylinder and $r = b$ the radius of its external surface. In Fig. 1(b), a cross-section of the structure is depicted with dimension and number of layers, which are mostly used in the forthcoming text.

Our goal is to optimize the axial and hoop stresses σ_{zz} and $\sigma_{\theta\theta}$ in the layers with respect to $\boldsymbol{\mu}_{zz}$ and $\boldsymbol{\mu}_{\theta\theta}$ i.e., with respect to free hoop and radial eigenstrains. From that one can derive optimal admissible stresses and prestrains/prestresses are the design parameters in the optimization problem.

2. Deterministic optimization of laminated cylinders

First we denote $\mathbf{m} = \boldsymbol{\mu} + \bar{\boldsymbol{\mu}}$ and $\mathbf{l} = \boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}}$, where $\mathbf{m}$ is the total eigenstrain tensor, $\bar{\boldsymbol{\mu}}$, is the given eigenstrain tensor (change of temperature, plastic strain, etc.), $\mathbf{l}$ is the total eigenstress tensor, $\bar{\boldsymbol{\lambda}}$ is the given eigenstress tensor (given prestress, relaxation stress, etc.). $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are the design variables in the optimization formulations. In what follows, the tensors are mostly written in the standard vector form, for simplicity.

In the first section of this chapter, basic considerations are introduced. Variational principles and optimization rules involving eigenparameters are defined in the second section. Note that there is a relation between eigenstrains $\mathbf{m}$ and eigenstresses $\mathbf{l}$, and it holds:

$$\mathbf{l} = -\mathbf{L}\mathbf{m}, \tag{1}$$

where $\mathbf{L}$ stands for elastic material stiffness matrix. From (1), it follows immediately that one can use either eigenstrains or eigenstresses, but not both in one formula. Simple examples show the properties of such optimizations in Appendix A, where they make clear the sense of the introduced deterministic optimization problems.

2.1. Computational model

The generalized plane strain is assumed (pseudo-three-dimensional problem) in the coordinate system $Ox_1x_2x_3$ and an arbitrary point in the domain is described as $\mathbf{x} = \{x_1x_2x_3\}$. It starts with the kinematical relations, which can be written as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j = 1,2,3. \tag{2}$$

Hooke's law for an anisotropic field is considered in the form ($\mathbf{M}$ is the compliance matrix, $\mathbf{M} = (\mathbf{L})^{-1}$):

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{M}(\mathbf{x})\boldsymbol{\sigma}(\mathbf{x}) + \mathbf{m}\mathbf{x}, \tag{3}$$

or alternatively

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{L}(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{x}) + \mathbf{l}(\mathbf{x}). \tag{4}$$

Substituting $\boldsymbol{\sigma}$ from (4) to the equations of equilibrium

$$\partial\boldsymbol{\sigma} + \mathbf{b} = 0, \tag{5}$$

where $\mathbf{b}$ is the volume weight vector, and then introducing displacement field $\mathbf{u}$ for strains from (2) leads to the Navier equations for unknown displacements:

$$\partial(\mathbf{L}\partial^{\mathrm{T}}\mathbf{u}) - \partial(\mathbf{L}\mathbf{m}) + \mathbf{b} = 0 \quad \text{in } \Omega, \tag{6}$$

or alternatively

$$\partial(\mathbf{L}\partial^{\mathrm{T}}\mathbf{u}) + \partial\mathbf{l} + \mathbf{b} = 0 \quad \text{in } \Omega, \tag{7}$$

where Ω is the domain of the body under consideration, T denotes transposition. In Eqs. (5)–(7) we denoted:

$$\partial = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix}. \tag{8}$$

The boundary conditions for (6) and (7) are defined as

$$\begin{aligned} &\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u, \\ &\mathbf{p} = \bar{\mathbf{p}} \quad \text{on } \Gamma_p, \\ &p_i = \sigma_{ij} n_j, \quad i,j = 1,2,3 \quad \text{on } \Gamma_p, \\ &\mathbf{n} = (n_1, n_2, n_3), \end{aligned} \tag{9}$$

where the barred quantities are prescribed, $\mathbf{n}$ is the unit outward normal.

In a cylindrical axisymmetric case, it is useful to introduce the cylindrical coordinates $0r\theta z$, see Fig. 1(a). For axisymmetric case, it holds ($\mathbf{x} = \{r, \boldsymbol{\theta}, z\}$):

$$\begin{aligned} &\varepsilon_r = \frac{\partial u_r}{\partial r}; \quad \varepsilon_\theta = \frac{u_r}{r}; \quad \varepsilon_z = \frac{\partial u_z}{\partial z} = \text{const.}; \\ &\varepsilon_{r\theta} = \varepsilon_{rz} = \varepsilon_{\theta z} = 0. \end{aligned} \tag{10}$$

The displacement vector and strain and stress tensors can be written in a contracted notation as

$$\begin{aligned} &\mathbf{u} = \{u_r, u_\theta, u_z\}^{\mathrm{T}}; \\ &\boldsymbol{\varepsilon} = \{\varepsilon_r, \varepsilon_\theta, \varepsilon_z\}^{\mathrm{T}}; \quad \boldsymbol{\sigma} = \{\sigma_r, \sigma_\theta, \sigma_z\}^{\mathrm{T}}. \end{aligned} \tag{11}$$

The only one equation of equilibrium holds (other equations are fulfilled identically):

$$\frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \sigma_r}{\partial r} = 0. \tag{12}$$

The results in this paper are attained for anisotropic materials ($M_{ij} = M_{ji}$). For axially orthotropic case $i,j = r,\theta,z$ the components of the compliance matrix may be expressed in terms of elasticity moduli and Poisson's numbers:

$$\mathbf{M} = \begin{bmatrix} M_{rr} & M_{r\theta} & M_{rz} \\ & M_{\theta\theta} & M_{\theta z} \\ \text{symm.} & & M_{zz} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_r} & -\frac{\nu_{r\theta}}{E_\theta} & -\frac{\nu_{rz}}{E_z} \\ -\frac{\nu_{\theta r}}{E_r} & \frac{1}{E_\theta} & -\frac{\nu_{\theta z}}{E_z} \\ -\frac{\nu_{zr}}{E_r} & -\frac{\nu_{z\theta}}{E_\theta} & \frac{1}{E_z} \end{bmatrix}. \tag{13}$$

Since the domain is denoted by Ω, the disjoint locally homogeneous but generally anisotropic subdomains (layers) are denoted as Ω_k, $k = 1, \ldots, n$, numbered from inner to outer layer. In each subdomain, the Hooke's law holds:

$$\boldsymbol{\varepsilon}^k(\mathbf{x}) = \mathbf{M}^k \boldsymbol{\sigma}^k(\mathbf{x}) + \mathbf{m}^k(\mathbf{x}), \quad k = 1, \ldots, n, \tag{14}$$

or alternatively

$$\boldsymbol{\sigma}^k(\mathbf{x}) = \mathbf{L}^k \boldsymbol{\varepsilon}^k(\mathbf{x}) + \mathbf{l}^k(\mathbf{x}), \quad k = 1, \ldots, n, \tag{15}$$

where $\mathbf{L}^k$ and $\mathbf{M}^k = (\mathbf{L}^k)^{-1}$ are, respectively, the material stiffness and the compliance in subdomains k, $\boldsymbol{\sigma}^k$ are the stresses and $\boldsymbol{\varepsilon}^k$ are the strains, $\mathbf{m}^k = \boldsymbol{\mu}^k + \bar{\boldsymbol{\mu}}^k$, $\bar{\boldsymbol{\mu}}^k$ are the prescribed eigenparameters creating an independent transformation field in Ω_k (due to change of temperature, current plastic state, creep, etc.), $\boldsymbol{\mu}^k$ are free design tensors (representing prestrain in our case). One can similarly split eigenstresses $\mathbf{l}^k = \boldsymbol{\lambda}^k + \bar{\boldsymbol{\lambda}}^k$ in Ω_k to the free and prescribed components.

The relations $\mathbf{l}^k = -\mathbf{L}^k \mathbf{m}^k$, $\mathbf{m}^k = -\mathbf{M}^k \mathbf{l}^k$, hold, so that (14) and (15) may be modified (no summation over k):

$$\begin{aligned} \boldsymbol{\varepsilon}^k(\mathbf{x}) &= \mathbf{M}^k \boldsymbol{\sigma}^k(\mathbf{x}) - \mathbf{M}^k \mathbf{l}^k(\mathbf{x}), \\ \boldsymbol{\sigma}^k(\mathbf{x}) &= \mathbf{L}^k \boldsymbol{\varepsilon}^k(\mathbf{x}) - \mathbf{L}^k \mathbf{m}^k(\mathbf{x}). \end{aligned} \tag{16}$$

The equilibrium between adjacent layers is fully discussed in Dvorak and Procházka (1996). We consider the interfacial conditions in the form of continuous radial displacements and the equilibrated radial tractions.

2.2. Extended Lagrange's principle for laminate cylinders

For completeness and also as an example we start with formulation of Lagrange's variational principle for the problem (6) and (7). From this principle, it follows that for the given $\bar{\mathbf{p}}$ on Γ_p, the volume weight $\mathbf{b}$ in Ω and the eigenstrain field $\mathbf{m}$ in Ω the solution $\mathbf{u} \in V$ of (6) and (7) is a minimum of the following energetic functional:

$$\begin{aligned} &\Pi_p(\mathbf{u}) = \frac{1}{2} \int_\Omega (\varepsilon - \mathbf{m})^{\mathrm{T}}(\mathbf{x}) \boldsymbol{\sigma}(\mathbf{x}) \, \mathrm{d}\Omega(\mathbf{x}) - \int_\Omega \mathbf{b}^{\mathrm{T}}(\mathbf{x}) \mathbf{u}(\mathbf{x}) \, \mathrm{d}\Omega(\mathbf{x}) - \int_{\Gamma_p} \bar{\mathbf{p}}^{\mathrm{T}}(\mathbf{x}) \mathbf{u}(\mathbf{x}) \, \mathrm{d}\Gamma(\mathbf{x}), \\ &V = \{\mathbf{u}; \mathbf{u} \text{ prescribed on } \Gamma_u\}. \end{aligned} \tag{17}$$

The functional (17) can be rewritten as

$$\begin{aligned} \Pi_p(\mathbf{u}, \boldsymbol{\mu}) = \frac{1}{2} \sum_{k=1}^{n} \int_{\Omega_k} [\boldsymbol{\varepsilon}^k(\mathbf{x}) - \boldsymbol{\mu}^k - \bar{\boldsymbol{\mu}}^k(\mathbf{x})]^{\mathrm{T}} \mathbf{L}^k [\boldsymbol{\varepsilon}^k(\mathbf{x}) - \boldsymbol{\mu}^k(\mathbf{x}) - \bar{\boldsymbol{\mu}}^k] \, \mathrm{d}\Omega_k(\mathbf{x}) - \int_\Omega \mathbf{b}^{\mathrm{T}}(\mathbf{x}) \mathbf{u}(\mathbf{x}) \, \mathrm{d}\Omega(\mathbf{x}) \\ - \int_\Gamma \bar{\mathbf{p}}^{\mathrm{T}}(\mathbf{x}) \mathbf{u}(\mathbf{x}) \, \mathrm{d}\Gamma_p(\mathbf{x}), \end{aligned} \tag{18}$$

where Ω_k is the domain of lamina number k. Additional constraint conditions have to be introduced because of independent behavior of particular laminas, which is not involved in the principle. From this point of view the above formulation is not practical. In what follows we use the ideas of transformation field analysis. They enable us to write the principle in a more compact way and the optimization is then

formulated in a comprehensive form. Using the principle (17) and (18) the formulation of optimization is principally cumbersome.

From (18), it is obvious that the additional conditions involve equilibrium and compatibility between layers. This is why we concentrate our attention on expressing physical quantities in the general laminated structure.

Let us denote by $\langle\cdot\rangle$ the volume averages of the functions. A simplification is introduced: the distribution of eigenparameters in one layer is uniform, i.e., $\boldsymbol{\mu}^k(\mathbf{x}) \equiv \langle\boldsymbol{\mu}^k\rangle$, $k = 1,\ldots,n$.

Suppose that in the first m laminas Ω_k, $k = 1,\ldots,m$ the eigenstrains $\boldsymbol{\mu}^k$ are independent. In each lamina we successively introduce mutually independent unit eigenstrains $\langle\mu_j^k\rangle^1$, $j = 1,\ldots,6$, $k = 1,\ldots,m$. From that, three components of displacements and six components of stresses and six components of strains are obtained in each lamina $i = 1,\ldots,n$, i.e., influence tensors are univalently defined by $\langle\mu_j^k\rangle^1$. $\mathbf{D}_k^i \equiv D_{\alpha kj}^i$ is the influence function tensor, the appropriate components of which in $D_{\alpha kj}^i$ are the strain ε_α^i responses to the unit impulses $\langle\mu_j^k\rangle^1 = 1$, $\mathbf{F}_k^i$ is the influence function tensor created in a similar manner for stress responses to the unit impulses $\langle\mu_j^k\rangle^1 = 1$, and eventually $\mathbf{U}_k^i$ is an influence function tensor for displacement responses.

The influence tensors depend on constant elastic moduli, which are influenced by given volume fractions of the phases in the individual layers. Then the volume fractions cannot affect the optimal prestressing.

The variables $\bar{\boldsymbol{\varepsilon}}^i(\mathbf{x})$, $\bar{\boldsymbol{\sigma}}^i(\mathbf{x})$, $\bar{\mathbf{u}}^i(\mathbf{x})$ in what follows are caused due to external loading and of prescribed eigenstrains. In this way one can split the elastic strain as

$$\varepsilon_\alpha^i(\mathbf{x}) = \varepsilon_\alpha^{-i}(\mathbf{x}) + \sum_{j=1}^{6}\sum_{k=1}^{m} D_{\alpha kj}^i(\mathbf{x})\langle\mu_j^k\rangle, \quad i = 1,\ldots,n,\ \ \alpha = 1,\ldots,6, \tag{19}$$

$$\boldsymbol{\varepsilon}^i(\mathbf{x}) = \bar{\boldsymbol{\varepsilon}}^i(\mathbf{x}) + \sum_{k=1}^{m} \mathbf{D}_k^i(\mathbf{x})\langle\boldsymbol{\mu}^k\rangle, \quad i = 1,\ldots,n. \tag{20}$$

Similarly, it holds the relation between the stresses and the eigenstrains:

$$\sigma_\alpha^i(\mathbf{x}) = \bar{\sigma}_\alpha^i(\mathbf{x}) + \sum_{j=1}^{6}\sum_{k=1}^{m} F_{\alpha kj}^i(\mathbf{x})\langle\mu_j^k\rangle, \quad i = 1,\ldots,n,\ \ \alpha = 1,\ldots,6, \tag{21}$$

or in a matrix form

$$\boldsymbol{\sigma}^i(\mathbf{x}) = \bar{\boldsymbol{\sigma}}^i(\mathbf{x}) + \sum_{k=1}^{m} \mathbf{F}_k^i(\mathbf{x})\langle\boldsymbol{\mu}^k\rangle, \quad i = 1,\ldots,n. \tag{22}$$

Eventually, the relation displacements and eigenstrains can be written as

$$u_\alpha^i(\mathbf{x}) = \bar{u}_\alpha^i(\mathbf{x}) + \sum_{j=1}^{6}\sum_{k=1}^{m} U_{\alpha kj}^i(\mathbf{x})\langle\mu_j^k\rangle, \quad i = 1,\ldots,n,\ \ \alpha = 1,\ldots,6, \tag{23}$$

or

$$\mathbf{u}^i(\mathbf{x}) = \bar{\mathbf{u}}^i(\mathbf{x}) + \sum_{k=1}^{m} \mathbf{U}_k^i(\mathbf{x})\langle\boldsymbol{\mu}^k\rangle, \quad i = 1,\ldots,n. \tag{24}$$

In every optimization problem, the admissible set of eigenstrains should obey some restricted values of components of eigestrain tensor. This is an impact of bearing capacity of fibers, exclusion of tensile stresses in fibers, etc. The set is denoted by $\mathcal{O}$ and defined as

$$\mathcal{O} = \{\boldsymbol{\mu};\ \langle\mu_\alpha^{l,i}\rangle \leqslant \langle\mu_\alpha^i\rangle \leqslant \langle\mu_\alpha^{u,i}\rangle,\ i = 1,\ldots,m,\ \alpha = 1,\ldots,6\} \tag{25}$$

for $\langle\mu_\alpha^{l,i}\rangle a\langle\mu_\alpha^{u,i}\rangle$, $i = 1,\ldots,n$, $\alpha = 1,\ldots,6$ given.

Let the matrices $\mathbf{F}^k$ and $\mathbf{D}^k$ and the barred quantities be given from previous computations. The barred variables follow from linear computation. They represent responses to external loading and to a given transformation filed. The matrices are influence tensors, their components are obtained from unit eigenstrain impulses in layers under assumption that there is no external load. Without loss of generality let us assume the volume weight to be zero. Then the following Lagrange's principle can be formulated.

Our aim is to find stationary point of Π_p on V and the class of functions $\langle \boldsymbol{\mu} \rangle \in \mathcal{O}$, where

$$\Pi_p(\mathbf{u}^k(\mathbf{x}), \langle \boldsymbol{\mu}^i \rangle) = \frac{1}{2} \sum_{k=1}^{n} \int_{\Omega k} \left[\bar{\boldsymbol{\varepsilon}}^k(\mathbf{x}) \sum_{j=1}^{m} \mathbf{D}_j^k(\mathbf{x}) \langle \boldsymbol{\mu}^j \rangle \right]^{\mathrm{T}} \mathbf{L}^k \left[\bar{\boldsymbol{\varepsilon}}^k(\mathbf{x}) + \sum_{j=1}^{m} \mathbf{D}_j^k(\mathbf{x}) \langle \boldsymbol{\mu}^j \rangle \right] \mathrm{d}\Omega_k(\mathbf{x}) - \int_{\Gamma} \bar{\mathbf{p}}^{\mathrm{T}}(\mathbf{x}) \mathbf{u}(\boldsymbol{\mu}; \mathbf{x}) \, \mathrm{d}\gamma. \tag{26}$$

As shown in Appendix A, where a simple laminated structure is treated, this problem leads to annihilation of stresses in the layers where the free eigenstrains are introduced. The above formulation plays very important role in numerical applications, as we have only one functional, which depends on two kinds of functions: the minimization with respect to displacement field solves elasticity problem and stationary point of eigenparameters gives the above-mentioned annihilation of stresses. The problem complies with Chapter 6 in Dvorak et al. (1999), where a problem of optimal strain distribution in the plane created by $0\theta z$ coordinations is sought. For example, solving this problem an optimal conditions for placement of insulation in the mentioned layer are attained. Assuming that the stresses in radial direction are negligible, from Hooke's law one obtains that not only stresses $(\sigma_\theta, \sigma_z)$, but also strains ε_θ, ε_z are equal to zero.

2.3. Optimal stress distribution

In this section our aim is to find the minimum Π_σ on a class of functions $\langle \boldsymbol{\mu} \rangle \in \mathcal{O}$ (see the previous section), where

$$\begin{aligned} \Pi_\sigma(\bar{\boldsymbol{\sigma}}^k(\mathbf{x}), \langle \boldsymbol{\mu}^i \rangle) &= \frac{1}{2} \sum_{k=1}^{n} \int_{\Omega_k} \boldsymbol{\sigma}^{\mathrm{T}}(\mathbf{x}) \boldsymbol{\sigma}(\mathbf{x}) \, \mathrm{d}\Omega_k(\mathbf{x}) \\ &= \frac{1}{2} \sum_{k=1}^{n} \int_{\Omega_k} \left[\bar{\boldsymbol{\sigma}}^k(\mathbf{x}) + \sum_{i=1}^{m} \mathbf{F}_i^k(\mathbf{x}) \langle \boldsymbol{\mu}^i \rangle \right]^{\mathrm{T}} \left[\bar{\boldsymbol{\sigma}}^k(\mathbf{x}) + \sum_{i=1}^{m} \mathbf{F}_i^k(\mathbf{x}) \langle \boldsymbol{\mu}^i \rangle \right] \mathrm{d}\Omega_k(\mathbf{x}). \end{aligned} \tag{27}$$

Since the latter problem is non-linear (the stresses and influence tensors are position-dependent), the quantities in the relations (21) and (22) can be substituted by their averages. Then the functional (27) reads:

$$\begin{aligned} \Pi_\sigma(\langle \boldsymbol{\sigma}^k \rangle, \langle \boldsymbol{\mu}^i \rangle) &= \frac{1}{2} \sum_{k=1}^{n} \int_{\Omega_k} \langle \boldsymbol{\sigma} \rangle^{\mathrm{T}} \langle \boldsymbol{\sigma} \rangle \, \mathrm{d}\Omega_k(\mathbf{x}) \\ &= \frac{1}{2} \sum_{k=1}^{n} \left[\langle \bar{\boldsymbol{\sigma}}^k \rangle - \sum_{i=1}^{m} \langle \mathbf{F}_i^k \rangle \langle \boldsymbol{\mu}^i \rangle \right]^{\mathrm{T}} \left[\langle \bar{\boldsymbol{\sigma}}^k \rangle - \sum_{i=1}^{m} \langle \mathbf{F}_i^k \rangle \langle \boldsymbol{\mu}^i \rangle \right] \text{ meas } \Omega_k, \end{aligned} \tag{28}$$

where ''meas'' means the measure of the argument. This simplification turns the above functional to a function of the free eigenstrains.

Differentiating Π_σ from (28) by $\langle \mu_\gamma^s \rangle$ leads to the following conditions (in detailed form):

$$\sum_{\alpha=1}^{6} \sum_{k=1}^{n} \left(\langle \bar{\sigma}_\alpha^k \rangle + \sum_{i=1}^{m} \sum_{\beta=1}^{6} \langle F_{i\alpha\beta}^k \rangle \langle \mu_\beta^i \rangle \right) F_{s\alpha\gamma}^k \quad \text{meas } \Omega_k = 0. \tag{29}$$

From (29) we get a system of linear algebraic equations:

$$A_{s\gamma i\beta}\langle\mu^i_\beta\rangle = b_{s\gamma}, \quad i,s = 1,\ldots,m, \quad \beta,\gamma = 1,\ldots,6, \tag{30}$$

where

$$A_{s\gamma i\beta} = \sum_{\alpha=1}^{6}\sum_{k=1}^{n}\langle F^k_{i\alpha\beta}\rangle\langle F^k_{s\alpha\gamma}\rangle \quad \text{meas } \Omega_k,$$

$$b_{s\gamma} = \sum_{\alpha=1}^{m}\sum_{k=1}^{n}\langle\bar{\sigma}^k_\alpha\rangle\langle F^k_{s\alpha\gamma}\rangle \quad \text{meas } \Omega_k.$$

The above optimization problem leads to maximization of bearing capacity of the laminated structure. This can be done by an appropriate prestrain in laminas. The way of introduction of such a prestrain in particular laminas and the process of fabrication is described in Dvorak et al. (1999) and Srinivas et al. (1999).

As discussed in Appendix A, the optimization of total energy deals with a comprehensive formulation of mechanical problem of laminated structures; the eigenparameters cause annihilation of stresses in layers where they are introduced. The optimization of stresses leads to uniform distribution of stresses through the thickness of the structure under consideration in that case when $m \leqslant n-1$ where n is the total number of layers and m is the number of layers with introduced eigenstrain.

Similarly to the restrictions (25) on eigenstrains, the admissible set can be stated for stresses (e.g., strengths). Then the lagrangian multipliers or penalty method can be applied. Using a mathematical treatment, it can be proved that the Hessian of the system is positively definite and consequently, Newton–Raphson method may be used for this non-linear system. This is out of scope of this paper.

2.4. Applications

Consider a balanced laminate hollow cylinder composed of 5 layers of equal thicknesses, which are arranged in a repeating $(0/\pm 60/90_2)$ lay-ups. The layers are made of AS4/3501-6 carbon–epoxy composite whose properties are the same for each layer [MPa]: $L_{rr} = 14240$, $L_{r\theta} = 5730$, $L_{rz} = 6506$, $L_{\theta\theta} = 94100.1$, $L_{\theta z} = 11699.6$, $L_{zz} = 41200.8$. The external hydrostatic loading is applied. Tractions $p_\theta = 20$ MPa in the θ-direction and $p_z = 20$ MPa in the z-direction. Inner radius is 4.5 m, outer radius is 5 m. The length of the cylinder is not important in our case.

Fig. 2 describes the distribution of stresses σ_θ and σ_z throughout the thickness in case of no eigenstrains.

Fig. 3 show the distribution of stresses due to unit eigenstrains applied in the third layer, for example. In these figures $\sigma^\theta_\theta \equiv \langle\sigma^\theta_\theta\rangle$ means the hoop stress resulting from $\langle\mu_\theta\rangle = 1$, $\sigma^z_z \equiv \langle\sigma^z_z\rangle$ stands for the axial stress

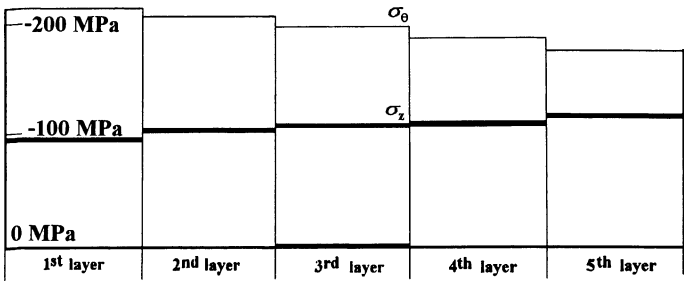

Fig. 2. Distribution of the stresses without eigenstrains.

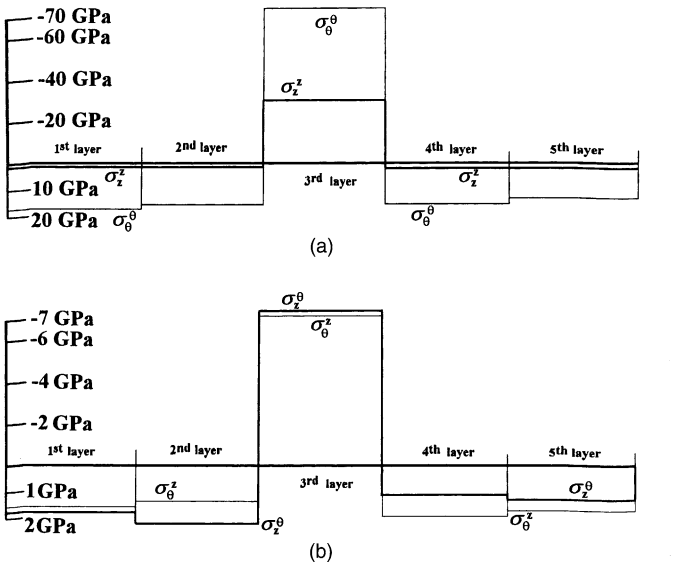

Fig. 3. Stress distribution due to unit eigenstrains.

resulting from $\langle \mu_z \rangle = 1$, $\sigma_\theta^z \equiv \langle \sigma_\theta^z \rangle$ stands for the axial stress due to $\langle \mu_z \rangle = 1$, and eventually $\sigma_z^\theta \equiv \langle \sigma_z^\theta \rangle$ stands for the hoop stress due to unit axial eigenstrain.

Applying Lagrange's principle, the eigenstrains belonging to the optimal state are: $\mu_\theta = -2.4324 \times 10^{-3}$ and $\mu_z = -2.864 \times 10^{-3}$. The distribution of stresses throughout the thickness of the structure is illustrated in Fig. 4 for both axial and hoop stresses.

The absolute maximum hoop stress appears in the first layer $\sigma_\theta^1 = -260.3026$ MPa. Because the problem is linear, the strength 700 MPa envisaged in the beginning of this paper is reached for $\mu_\theta = -6.5412 \times 10^{-3}$ and $\mu_z = -7.702 \times 10^{-3}$. If the prestraining is higher than the above values then the bearing capacity would be exceeded.

The previous results can partly be used in the optimization of stresses over the entire structure. Studying the above described laminate hollow cylinder with 5 equidistant layers yields a couple of possibilities about what and how to pose the optimization problems. One can optimize either the hoop stress or the axial stress or both. As design parameters can serve either the hoop eigenstress or the axial eigenstress or both. To start with, we concentrate our attention on optimization of the hoop stresses with the hoop eigenstrains as the design parameters. For a simple structure it is proven in Appendix A that the number of design parameters

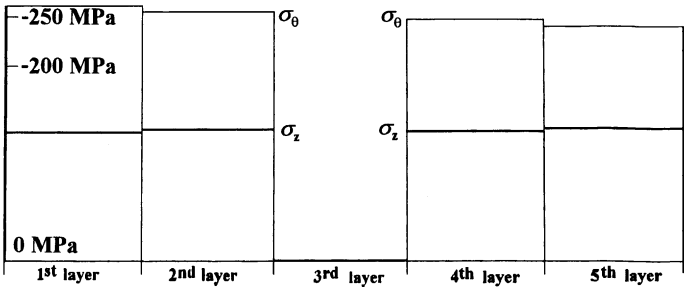

Fig. 4. Distribution of the stresses with eigenstrains introduced in the layer 3.

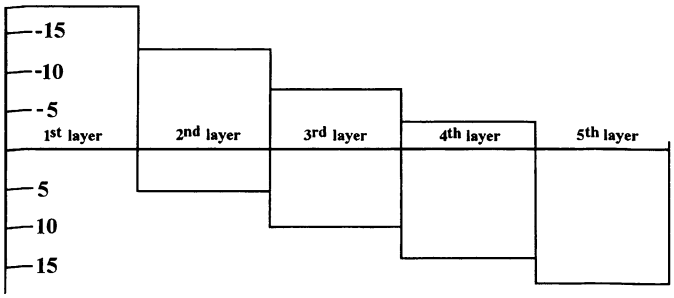

Fig. 5. Distribution of the hoop eigenstrains $\mu_\theta \times 10^{-5}$ for the optimal hoop stresses.

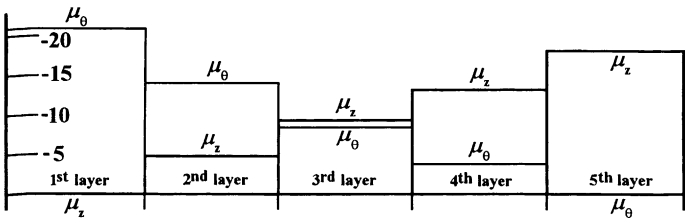

Fig. 6. Distribution of the hoop eigenstrains $\mu_\theta \times 10^{-5}$ and $\mu_z \times 10^{-5}$ for the optimal stresses.

m is restricted to the number of layers n minus one. It means that if the hoop stresses are optimized, at most $n-1$ eigenparameters can be introduced, for example. If both hoop and axial stresses are optimized, the number of both hoop and axial eigenstrains cannot exceed $n-1$.

In Fig. 5, distributions of the hoop eigenstrains are shown in the problem of optimization of the hoop stresses. Two cases are considered: first, the fifth layer is dropped out and then the first layer is excluded from computation. It is obviously seen from the figure that a monotonous character of distributions of the hoop stresses is reached. Consequently, the negative eigenstrains are attained in the first case and also in the cases where in abandoned layer a negative high enough eigenstrain is prescribed. The distribution of the eigenparameters is almost uniform in that case.

Starting with this experience, a special choice of eigenstrains is introduced when optimizing both hoop and axial stresses. The result is shown in Fig. 6. The optimal hoop stress is −200.00 MPa and the optimal axial stress is −105.26 MPa. Taking into account the optimal prestrain to be introduced into the cylinder layers with a high accuracy (the fibers are carefully wound up, low matrix porosity is assumed, etc.), the strength of 700 MPa can allow 3.5 times higher external hydraulic pressure ($700/200 = 3.5$).

3. Statistic properties of the cost functionals

This section deals with the behavior of a functional serving for optimization of eigenstrains with stochastically perturbed and correlated values in a laminated cylindrical structure from composite materials. For expressing the relation between stresses and eigenstrains, the equations derived in the previous sections are used, namely (21) and (22). As a model, the optimal eigenstrains are sought, for which the cost functional Π_σ in (27) attains its minimum. The eigenstrains have now stochastically perturbed and correlated values.

3.1. Basic considerations

As in the previous sections, consider a laminate structure, which in undeformed state occupies the domain Ω. Again, the layers are represented by Ω_i, $i = 1, \ldots, n$, n is the number of layers. The geometry and denotation of the problem is the same as that depicted in Fig. 1(a) and (b).

For simplicity, assume that there is an external axisymmetric hydrostatic loading p along the entire boundary of the structure. The stress $\boldsymbol{\sigma}^k$ at an arbitrary point $\mathbf{x}$ of the domain Ω^k may be expressed as a superposition of the stress $\bar{\boldsymbol{\sigma}}^k$ at $\mathbf{x} \in \Omega^k$ due to the external loading $\mathbf{p}$ and a linear hull of the volume average eigenstrains $\langle \boldsymbol{\mu}^i \rangle$ applied in the layers Ω^i, $i = 1, \ldots, m$, see (21) and (22).

The optimization problem then starts with the functional Π_σ, which is defined in (28), for simplicity.

Let us consider deviations $\mathbf{d}_j$ from $\langle \boldsymbol{\mu} \rangle_j$; the latter are computed from the deterministic problem. From this, the functional Π_σ in (28) turns to the functional

$$\begin{aligned}\Phi(\langle \boldsymbol{\mu}^i \rangle + \mathbf{d}_i) &= \frac{1}{2} \sum_{k=1}^{n} \langle \boldsymbol{\sigma} \rangle^{\mathrm{T}} \langle \boldsymbol{\sigma} \rangle \quad \text{meas } \Omega_k \\ &= \frac{1}{2} \sum_{k=1}^{n} \left[\langle \bar{\boldsymbol{\sigma}}^k \rangle + \sum_{i=1}^{m} \langle \mathbf{F}_i^k \rangle (\langle \boldsymbol{\mu}^i \rangle + \mathbf{d}_i) \right]^{\mathrm{T}} \left[\langle \bar{\boldsymbol{\sigma}}^k \rangle + \sum_{i=1}^{m} \langle \mathbf{F}_i^k \rangle (\langle \boldsymbol{\mu}^i \rangle + \mathbf{d}_i) \right] \quad \text{meas } \Omega_k. \end{aligned} \tag{31}$$

In order not to complicate the index notation, the relations needed for stochastic estimates are derived only for the hoop direction. The values of the hoop stresses are principally greater than the radial stresses, which are completely excluded from the optimization. Involving the axial stresses would not make any trouble. Consequently, the axial eigenstrains are excluded from the next consideration without loss of generality. This means that instead of $\langle \boldsymbol{\sigma}^i \rangle$ we write $\langle \sigma_{\theta\theta}^i \rangle$ and instead of $\langle \boldsymbol{\mu}^i \rangle$, $\langle \mu_{\theta\theta}^i \rangle$ is used. We recall again that the generalization to both axial and hoop directions is easy.

The functional (31) can then be simplified as

$$\begin{aligned}\Phi(\langle \mu_{\theta\theta}^i \rangle + d_i) &= \frac{1}{2} \sum_{k=1}^{n} \langle \sigma_{\theta\theta}^k \rangle^{\mathrm{T}} \langle \sigma_{\theta\theta}^k \rangle \quad \text{meas } \Omega_k \\ &= \frac{1}{2} \sum_{k=1}^{n} \left[\langle \bar{\sigma}_{\theta\theta}^k \rangle + \sum_{i=1}^{m} \langle F_i^k \rangle (\langle \mu_{\theta\theta}^i \rangle + d_i) \right]^{\mathrm{T}} \left[\langle \bar{\sigma}_{\theta\theta}^k \rangle + \sum_{i=1}^{m} \langle F_i^k \rangle (\langle \mu_{\theta\theta}^i \rangle + d_i) \right] \quad \text{meas } \Omega_k, \end{aligned} \tag{32}$$

where d_i are now deviations of the design hoop eigenstrains.

In what follows we assume that the deviations of the design parameters from their nominal values are not too large, they are centered and moreover, the cost functional in its domain is a continuous and sufficiently smooth function of these parameters. As we study the problem in some distance from the stationary point, the linear approximation is not sufficient and we have to study the original quadratic functional (32).

In the neighborhood of a selected point (the deterministic optimal solution) $\langle \hat{\mu}_{\theta\theta}^k \rangle$, the functional (32) at the point $\langle \mu_{\theta\theta}^k \rangle$ may be expressed in Taylor's series as

$$\Phi(\langle \mu_{\theta\theta}^i \rangle + d_i) = \Phi(\langle \hat{\boldsymbol{\mu}}_{\theta\theta} \rangle) + \sum_{i=1}^{m} s_i d_i + \sum_{i,j=1}^{m} q_{ij} d_i d_j, \tag{33}$$

where $\Phi(\langle \hat{\boldsymbol{\mu}}_{\theta\theta} \rangle)$ is the value of the functional in case of nominal values or the mathematical means of the parameters, in our study deterministic optimal solution; $\mathbf{s} = |s_i|$, $\mathbf{q} = |q_{ij}|$, the vector or matrix of the parameters of the approximate function,

$$s_i = \frac{\partial \Phi(\langle \hat{\boldsymbol{\mu}}_{\theta\theta} \rangle)}{\partial \langle \mu_{\theta\theta}^i \rangle},$$

$$q_{ij} = \frac{1}{2}\frac{\partial^2 \Phi(\langle \hat{\boldsymbol{\mu}}_{\theta\theta} \rangle)}{\partial \langle \mu^i_{\theta\theta} \rangle \partial \langle \mu^j_{\theta\theta} \rangle}, \quad i,j = 1,\dots m,$$

n is the number of layers; m, the number of statistical variables (parameters), here the number of independent eigenstrains in the set $\langle \mu^i_{\theta\theta} \rangle$, $i = 1,\dots,n$.

From (32), it follows that

$$\begin{aligned} s_i &= 2\sum_{\alpha=1}^{n} \langle F^\alpha_i \rangle \left[\langle \bar{\sigma}^\alpha_{\theta\theta} \rangle + \sum_{\beta=1}^{m} \langle F \rangle^\alpha_\beta \langle \hat{\mu}^\beta_{\theta\theta} \rangle \right] \text{ meas } \Omega_\alpha, \\ q_{ij} &= \sum_{\alpha=1}^{n} \langle F^\alpha_i \rangle \langle F^\alpha_j \rangle \quad \text{meas } \Omega_\alpha. \end{aligned} \tag{34}$$

First, the mathematical mean of the functional is computed, i.e., in (33) apply the operator of the mathematical mean $\mathbf{E}\{\cdot\}$:

$$\mathbf{E}\{\Phi(\langle \boldsymbol{\mu}_{\theta\theta} \rangle)\} = \Phi(\langle \hat{\boldsymbol{\mu}}_{\theta\theta} \rangle) + \sum_{i,j=1}^{m} q_{ij} K_{ij}, \tag{35}$$

where $s_i = 0$, $i = 1,\dots,n$, and K_{ij} is the correlation of the parameters with perturbations; square matrix $n \times n$.

A natural requirement now is to find such a mean $\mathbf{E}\{\Phi(\langle \boldsymbol{\mu}_{\theta\theta} \rangle)\}$ that does not exceed the value of $\Phi(\mathbf{0})$, i.e., the prestraining does not worsen the stress state in the considered cylinder prior to introduction of the eigenstrains. It is worth noting that for non-correlated deviations the difference

$$\Phi(\mathbf{0}) - \Phi(\langle \boldsymbol{\mu}_{\theta\theta} \rangle + d_i) > 0. \tag{36}$$

In the case of *correlated* deviations one can proceed as follows.

As is supposed in the beginning of this chapter, a model example is considered for the functional Π_σ. For this problem, it holds:

$$\mathbf{E}\{\Phi(\langle \boldsymbol{\mu}_{\theta\theta} \rangle)\} = \Phi(\langle \hat{\boldsymbol{\mu}}_{\theta\theta} \rangle) + \sum_{i=1}^{n} \sum_{k,l=1}^{m} \langle F^i_k \rangle K_{kl} \langle F^i_l \rangle \quad \text{meas } \Omega_i. \tag{37}$$

Note that if the perturbances of parameters are statistically independent, (37) becomes

$$\mathbf{E}\{\Phi(\langle \boldsymbol{\mu}_{\theta\theta} \rangle)\} = \Phi(\langle \hat{\boldsymbol{\mu}}_{\theta\theta} \rangle) + \sum_{i=1}^{m} q_{ii} D_{ii}, \tag{38}$$

where D_{ii} is the diagonal matrix $m \times m$ of parameter dispersions.

If the functional attains its minimum at the point $\langle \hat{\boldsymbol{\mu}}_{\theta\theta} \rangle$ (deterministic optimal point), each q_{ii} is positive. As, in the same time, the dispersions D_{ii} are also positive, it has to hold:

$$\mathbf{E}\{\Phi(\langle \boldsymbol{\mu}_{\theta\theta} \rangle)\} > \Phi(\langle \hat{\boldsymbol{\mu}}_{\theta\theta} \rangle). \tag{39}$$

From that it follows that any uncertainty in the parameters may only worsen the optimal state, as the mathematical mean point of the process $\Phi(\langle \boldsymbol{\mu}_{\theta\theta} \rangle)$ is always greater than its value at the optimal point $\langle \hat{\boldsymbol{\mu}}_{\theta\theta} \rangle$.

Now, for correlated values of the deviations of eigenstrains it has to hold instead of (36):

$$\Phi(\mathbf{0}) - \mathbf{E}\{\Phi(\langle \boldsymbol{\mu}_{\theta\theta} \rangle)\} > 0. \tag{40}$$

From the latter inequality one can obtain certain estimates on the admissible values of deviations d_i. In the examples the correlation matrix of the perturbations is selected in the most used form, namely in the form of distance function. An alternative exponential function completes our study.

3.2. Application

The cylindrical laminate composite structure identical with that considered in the previous sections, which is situated in the coordinate system $0r\theta z$, is considered. Our aim is to find the reasonable distribution of deviations d_i from the optimized eigenstrains in the individual layers. Recall that the outer hydraulic compressive load $p = -20$ MPa, the thickness of the structure is 50 cm, the outer radius is 4 m, and the structure is built up from 5 equidistant layers. The stiffness coefficients of the AS4/3501-6$(O_{12}/90_{38})_S$ laminate are: $L_{rr} = 14.240$ GPa, $L_{r\theta} = 5.73$ GPa, $L_{rz} = 6.506$ GPa, $L_{\theta\theta} = 112.847$ GPa, $L_{\theta z} = 5.73$ GPa, $L_{zz} = 49.792$ GPa.

As the hoop stresses possess the prevailing meaning in comparison with the other components of the stress tensor, only these quantities are taken into account. The hoop eigenstrains are the only arguments of the cost functional in the optimization problems.

The generalized plane strain (Dvorak and Procházka, 1993), is assumed in the computation. Taking use of the optimization procedure for the minimization of the cost functional (3) with the hoop eigenstrains applied in the layers $1, \ldots, 4$, the average compressive hoop stress in the cylinder attains the value of -200 MPa and $\Phi(\hat{\boldsymbol{\mu}}) = 447.20$, while $\Phi(\mathbf{0}) = 447.40$. The initial average hoop stresses in the layers are computed as

$$\langle \bar{\boldsymbol{\sigma}}_{\theta\theta} \rangle = (-208.90 \quad -203.68 \quad -199.28 \quad -195.60 \quad -192.56)^{\mathrm{T}} \quad \text{MPa}$$

and the resulting optimal eigenstrains are

$$\langle \hat{\boldsymbol{\mu}}_{\theta\theta} \rangle = (-1.864 \quad -1.274 \quad -0.772 \quad -0.350)^{\mathrm{T}} \times 10^{-4}.$$

The influence matrix (tensor of the second order) is

$$\langle \mathbf{F} \rangle = \begin{bmatrix} -7.235 & 1.926 & 1.909 & 1.898 & 1.893 \\ 1.885 & -7.293 & 1.874 & 1.864 & 1.858 \\ 1.829 & 1.835 & -7.338 & 1.836 & 1.831 \\ 1.781 & 1.787 & 1.798 & -7.370 & 1.809 \\ 1.740 & 1.746 & 1.757 & 1.772 & -7.391 \end{bmatrix} \times 10^4$$

while the coefficients of approximation of the cost functional are given in the following table:

$q_{ij}[\times 10^9]$			
18.909	−5.084	−4.995	−4.938
−5.084	19.526	−5.028	−4.972
−4.995	−5.028	20.084	−5.034
−4.938	−4.972	−5.034	20.580

Let us assume that the correlation matrix be given by virtue of the distance function: $K_{ij} = K_0|r_i - r_j|$, where r_i is the radius of the central surface of the layer i. Then $K_{ij} = 0.1\ K_0|i - j|$, and K_0 is some real, which will be calculated from the condition (40). Easy calculation shows that (40) can be written as

$$\Phi(\mathbf{0}) - (\Phi(\langle \hat{\boldsymbol{\mu}}_{\theta\theta} \rangle + K_0 \times 1.9891 \times 10^1 0))0.$$

This implies that in our example $K_0 < 10^{-11}$.

Because in fifth layer no prestrain is introduced, $d_5 = 0$. Comparing the nature of formulas (33) and (35), the deviations depend on K_0 and the correlation function (matrix). One possible solution is $d_1 = d_2 = d_3 = d_4 = 10^{-6}$. Another admissible solution is $d_1 = 11.63 \times 10^{-7}$, $d_2 = 8.52 \times 10^{-7}$, $d_3 = 5.68 \times 10^{-7}$,

$d_4 = 2.84 \times 10^{-7}$. The first deviations follow from the assumption of their uniform distribution and the second one from the linear assumption.

Let us assume another correlation matrix: $K_{ij} = K_0 \exp|r_i - r_j|$. Then $K_{ij} = K_0 \exp(0.1|i-j|)$, and $K_0 < 10^{-12}$, using the same approach as before. For this correlation matrix we get even stronger requirement on values of the deviations. For the uniform distribution it holds: $d_1 = d_2 = d_3 = d_4 = 4.52 \times 10^{-7}$, and the linear distribution provides: $d_1 = 5.28 \times 10^{-7}$, $d_2 = 3.96 \times 10^{-7}$, $d_3 = 2.64 \times 10^{-7}$, $d_4 = 1.32 \times 10^{-7}$. In this case, more natural and accurate distribution appears to be an exponential one: $d_1 = 3.7 \times 10^{-7}$, $d_2 = 1.37 \times 10^{-7}$, $d_3 = 0.51 \times 10^{-7}$, $d_4 = 0.19 \times 10^{-7}$.

4. Conclusions

In this paper both deterministic and stochastic approaches to optimal prestresses of cylindrical laminated composite structures are discussed. The generalized plane strain is used as a pseudo-three-dimensional numerical model for solving the problem. The optimization of stresses with the eigenstrains serving as design parameters is explained on a simple example in Appendix A. First, hoop directions are preferred, while next also both hoop and axial directions are taken into account. The radial stresses and eigenstrains are neglected. The stochastic approach enables one to assess admissible deviations in prestressing to reach at least the same bearing capacity than that obtained from the model without prestressing.

Acknowledgement

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Appendix A

In this paper it has been shown that a minimization of the variance of stresses in laminate cylinder leads to unified distribution of stresses along the thickness of a cross-section of the structure, in the case $m = n - 1$, i.e., if the number m of prestressed layers is less than the number of all layers n minus one. There is a unique solution $\boldsymbol{\lambda}$, if the number of components λ^k, $k = 1, \ldots, m$ fulfills the inequality $m \leqslant n - 1$.

We prove the validity of the above-mentioned assertion on a simple beam structure (axial cut from a laminate cylinder), generally with different layer thickness and different material properties—see Fig. 7. At the left edge the beam is clamped while at the right edge $x = L$ (L is the length of the beam) a uniformly distributed load with a resultant F is applied. Only axial movement is admissible, i.e., both strain and stress occur exclusively in axial direction. The other components of both stress and strain tensors vanish.

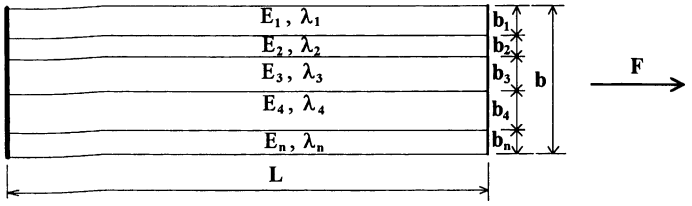

Fig. 7. Geometry of the beam.

Assumption. Generalized plane strain is considered, i.e., $\varepsilon^1 = \varepsilon^2 = \cdots = \varepsilon^n = \varepsilon = \text{const}$. From that $u(x) = L\varepsilon(x)$.

Under this assumption the Hooke's law in each layer reads:

$$\sigma^i = E^i(\varepsilon - \mu^i), \quad \mu^i = -\frac{\lambda^i}{E^i}, \quad i = 1, 2, \ldots, n, \tag{A.1}$$

so that

$$\sigma^i = E^i\varepsilon + \lambda^i, \quad i = 1, 2, \ldots, n. \tag{A.2}$$

Denote N^i the force in the layer i. The balance condition yields:

$$N^1 + N^2 + \cdots + N^n = F \Rightarrow \sum_{j=1}^{n} b^j E^j \varepsilon + \sum_{j=1}^{n} b^j \lambda^j = F. \tag{A.3}$$

The overall deformation and the stresses in the layers then are

$$\varepsilon = \frac{F - \sum_{j=1}^{n} b^j \lambda^j}{\sum_{j=1}^{n} b^j E^j} \quad \sigma^i = E^i \frac{F - \sum_{j=1}^{n} b^j \lambda^j}{\sum_{j=1}^{n} b^j E^j} + \lambda^i, \quad i = 1, 2, \ldots, n. \tag{A.4}$$

The internal energy E_{int} is defined as (see chapter on Lagrange's principle):

$$E_{\text{int}} = \frac{1}{2}\int_\Omega W^2 \,\mathrm{d}\Omega = \frac{1}{2}\sum_{i=1}^{n}\int_\Omega E^i(\varepsilon - \mu^i)^2 \,\mathrm{d}\Omega = \frac{L}{2}\sum_{i=1}^{n} E^i b^i \left(\varepsilon + \frac{\lambda^i}{E^i}\right)^2 = \frac{L}{2}\sum_{i-1}^{n} b^i \frac{(\sigma^i)^2}{E^i}. \tag{A.5}$$

For the sake of completeness note that the complementary internal energy E^*_{int} can be written as

$$E^*_{\text{int}} = \frac{1}{2}\int_\Omega (W^*)^2 \,\mathrm{d}\Omega = \frac{1}{2}\sum_{i=1}^{n}\left(2\sigma^i\mu^i + \frac{(\sigma^i)^2}{E^i}\right) = \frac{L}{2}\sum_{i=1}^{n}\frac{b^i}{E^i}[(\sigma^i - \lambda^i)^2 - (\lambda^i)^2]. \tag{A.6}$$

The meaning of the extended densities W and W^* of the internal energies in 1D is shown in Fig. 8.

Recast stresses from (A.4) as

$$\sigma^i = \bar{\sigma}^i + \sum_{j=1} D_{ij}\lambda^i, \quad \bar{\sigma}^i = \frac{E^i F}{\sum_{k=1}^{n} b^k E^k} \quad D_{ij} = \delta_{ij} - \frac{b^j E^i}{\sum_{k=1}^{n} b^k E^k}, \tag{A.7}$$

where δ_{ij} is Kronecker's delta, $\bar{\sigma}^i$ is the stress only due to the external load F and D_{ij} is the influence tensor (cf. Dvorak and Procházka, 1996).

Let us see what happens when we optimize the structure of the beam in the sense of minimum of the variance of stresses. Then the functional Π_σ is

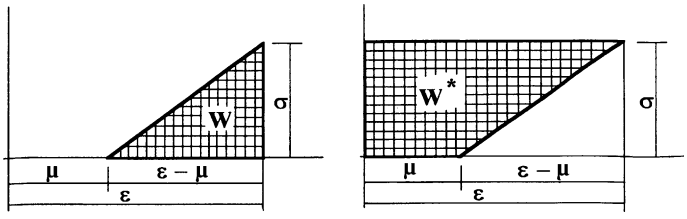

Fig. 8. Primary and dual densities of internal energies.

$$\Pi_\sigma = \int_\Omega \sigma(x)^2 \,\mathrm{d}x = L\sum_{i=1}^{n} b^i(\sigma^i)^2 = L\sum_{i=1}^{n} b^i\left(\bar{\sigma}^i + \sum_{j=1}^{n} D_{ij}\lambda^j\right)^2. \tag{A.8}$$

It is worth noting that if the stiffnesses of layers b^i/E^i are uniform for every admissible i then the functional to be minimized is exactly equal to internal energy in Lagrange's principle.

The condition of minimum of Π_σ leads to a system of linear equations for unknown λ^j:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} b^i D_{ij} D_{ik}\lambda^j = -\sum_{i=1}^{n} b^i \bar{\sigma}^i D_{ik}, \quad k = 1,\ldots,n. \tag{A.9}$$

For $n = 2$ we obtain a system of equations:

$$\begin{bmatrix} b^1D_{11}D_{11} + b^2D_{21}D_{21} & b^1D_{11}D_{12} + b^2D_{21}D_{22} \\ b^1D_{12}D_{11} + b^2D_{21}D_{22} & b^1D_{12}D_{12} + b^2D_{22}D_{22} \end{bmatrix}\begin{pmatrix}\lambda^1\\ \lambda^2\end{pmatrix} = \begin{bmatrix} b^1D_{11} & b^2D_{21} \\ b^1D_{12} & b^2D_{22} \end{bmatrix}\begin{bmatrix} D_{11} & D_{21} \\ D_{12} & D_{22} \end{bmatrix}\begin{pmatrix}\lambda^1\\ \lambda^2\end{pmatrix}$$
$$= -\begin{bmatrix} b^1D_{11} & b^2D_{21} \\ b^1D_{12} & b^2D_{22} \end{bmatrix}\begin{pmatrix}\bar{\sigma}^1\\ \bar{\sigma}^2\end{pmatrix}. \tag{A.10}$$

It can easily be proved that

$$\det\begin{bmatrix} D_{11} & D_{21} \\ D_{12} & D_{22} \end{bmatrix} = 0, \tag{A.11}$$

consequently, the eigenstresses are mutually dependent. It can also be shown that the rank of this matrix is equal to 1. This implies that there is an infinite number of solutions. This result complies with the physical meaning of the eigenstresses: we cannot prestress the structure in such a way that the overall stresses disappear because of equilibrium of internal and external forces.

System (A.11) may now be recast as

$$\begin{bmatrix} b^1D_{11} & b^2D_{21} \\ b^1D_{12} & b^2D_{22} \end{bmatrix}\begin{pmatrix}\bar{\sigma}^1 + D_{11}\lambda^1 + D_{12}\lambda^2\\ \bar{\sigma}^2 + D_{21}\lambda^1 + D_{22}\lambda^2\end{pmatrix} = \begin{bmatrix} b^1D_{11} & b^2D_{21} \\ b^1D_{12} & b^2D_{22} \end{bmatrix}\begin{pmatrix}\sigma^1\\ \sigma^2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}. \tag{A.12}$$

Substituting from (A.7) to (A.12) yields:

$$\begin{bmatrix} b^1b^2E^2 & -b^2b^2E^2 \\ -b^1b^2E^1 & b^1b^2E^1 \end{bmatrix} \tag{A.13}$$

and the first column is obviously linearly dependent on the second column. From this assertion immediately follows that stresses are the same for the arbitrary areas, thicknesses, and material parameters of the layers of the structure. The magnitude of the stresses follows from the equilibrium between the external and internal forces.

For $n > 2$ we generally get (A.9). It is sufficient to prove that

$$\det\begin{bmatrix} D_{11} & D_{12} & ,\ldots, & D_{1n} \\ \cdot & \cdot & ,\ldots, & \cdot \\ \cdot & \cdot & ,\ldots, & \cdot \\ \cdot & \cdot & ,\ldots, & \cdot \\ D_{n1} & D_{n2} & ,\ldots, & D_{nn} \end{bmatrix} = 0. \tag{A.14}$$

This can be done by mathematical induction. Similarly, the rank of the above matrix is $n - 1$.

For $n > 2$ a similar nature of Eq. (A.12) for $n > 2$ holds valid as

$$\sum_{j1}^{n} b^j D_{ji} \sigma^j = 0, \quad \text{for each } i = 1, \ldots, n. \tag{A.15}$$

Moreover, since it holds

$$b^i D_{ii} = -\sum_{j \neq i} b^i D_{ji}, \quad \text{for each } i = 1, \ldots, n \tag{A.16}$$

it immediately follows that according to (A.9) and (A.14), i.e., in the case that the number m of $\{\lambda^i\}_{i=1}^m$ is equal to $n-1$, then $\sigma^1 = \sigma^2 = \cdots = \sigma^n$.

From the above study it easily follows that even in one layer the eigenstress (or alternatively eigenstrain) can possess certain given value; the prestressing in the other layers are then calculated from optimization. Moreover, the transformation field can be prescribed together with one free eigenparameter and still optimal distribution of stresses is computed from minimization of the functional Π_σ in (A.8), which is equivalent to that from the second chapter. The stresses in the individual layers do not change in value although the prestressing can change.

Furthermore, we are going to show another one type of functional, which has been discussed in the previous text. It leads to zero stresses in selected layers, and subsequently it suppresses strains in these layers. The reason of this optimization is the same as in the sixth chapter of the publication by Dvorak et al. (1999).

This can be of interest to designers, who use eigenparameters as an optimization tool. The above denoted quantities and derived formulas will also be used in what follows.

The total strain energy, following Lagrange's principle, in our case is

$$\Pi_1 = E_{\text{int}} - E_{\text{ext}} \to \text{minimum}, \tag{A.17}$$

where E_{ext} is the external energy defined as

$$E_{\text{ext}} = Fu = FL\varepsilon = FL\frac{F - \sum_{j=1}^{n} b^j \lambda^j}{\sum_{j=1}^{n} b^j E^j}. \tag{A.18}$$

After differentiating Π_1 by λ^k, and with respect to (A.7) and (A.18) gives:

$$\sum_{i=1}^{n} \frac{b^i}{E^i} \sigma^i D_{ik} + F\frac{b^k}{\sum_{j=1}^{n} b^j E^j} = 0. \tag{A.19}$$

Let us rearrange the first term of the left hand side as

$$\sum_{i=1}^{n} \frac{b^i}{E^i} \sigma^i D_{ik} = \sum_{i=1}^{n} \frac{b^i}{E^i} \sigma^i \left(\delta_{ik} - \frac{E^i b^k}{\sum_{j=1}^{n} b^j E^j}\right) = \frac{b^k}{E^k}\sigma^k - \frac{\sum_{i=1}^{n} b^i b^k \sigma^i}{\sum_{i=1}^{n} b^i E^i} = \left(\frac{\sigma^k}{E^k} - \frac{\sum_{i=1}^{n} b^i \sigma^i}{\sum_{i=1}^{n} b^i E^i}\right) b^k. \tag{A.20}$$

Since $\sum_{i=1}^{n} b^i \sigma^i = F$, from this and (A.19) we obtain that in the layer k the stress is equal to zero. Similarly, if into m layers eigenparameters are introduced, stress-free state is obtained in these layers. As the equilibrium condition must be obeyed, the number of free eigenstresses cannot exceed $n-1$ again.

Conclusion: Minimization of the variance of stresses leads to a unification of stresses along the thickness of the beam, in the case $m = n-1$. There is a unique solution $\boldsymbol{\lambda}$ when the number of components λ^k, $k = 1, \ldots, m$ fulfills $m \leqslant n-1$.

Minimization of total strain energy with respect to the components of tensor $\boldsymbol{\lambda}$ leads to zero stresses at each layer where the eigenstress (or eigenstrain) is introduced. The admissible number of free eigenstresses is $n-1$ again.

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